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On hamiltonian colorings of graphs

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Abstract

A hamiltonian coloring of a connected graph G of order n is an assignment c of colors (positive integers) to the vertices of G such that $|c(u)-c(v)|+D(u,v)\geqslant n-1$ for every two distinct vertices u and v of G, where $D(u,v)$ is the length of a longest u–v path in G. For a hamiltonian coloring c, $\mathrm{hc}(c)$ is the largest color assigned to a vertex of G; while the hamiltonian chromatic number $\mathrm{hc}(G)=\min\{\mathrm{hc}(c)\}$ over all hamiltonian colorings c of G. The circumference $\mathrm{cir}(G)$ of a graph G is the length of a longest cycle in G. A lower bound for $\mathrm{cir}(G)$ is given in terms of the number of vertices that receive colors between two specified colors in a hamiltonian coloring of G. As a consequence of this result, it follows that if there exists a hamiltonian coloring of a connected graph G of order $n\geqslant 3$ such that at least $(n+2)/2$ vertices of G are colored the same, then G is hamiltonian. Also, if there exists a hamiltonian coloring of a connected graph G of order $n\geqslant 4$ such that at least $(n+2)/2$ vertices of G are colored with one of two consecutive colors, then $\mathrm{cir}(G)\geqslant n-1$. Furthermore, it is shown that if G is a connected graph of order $n\geqslant 4$ with $2\leqslant \mathrm{hc}(G)\leqslant n-1$, then $\mathrm{cir}(G)\geqslant n+2-\mathrm{hc}(G)$. Moreover, if G is a connected graph of order $n\geqslant 5$ that is not a star, then $\mathrm{hc}(G)\leqslant (n-2)^2-1$.

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1. Introduction

Two of the major areas in graph theory are colorings and the study of longest paths and cycles. Within the second area is hamiltonian graph theory, which includes a number of theorems that give sufficient conditions for graphs to contain hamiltonian cycles or cycles of some prescribed length. Another major topic of study in hamiltonian graph theory is hamiltonian-connected graphs (graphs containing a hamiltonian u–v path for every pair u, v of distinct vertices). It is the goal of this paper to study a connection between these two areas.

A proper or standard coloring of a connected graph G of order n and diameter d, can be defined as a function c from $V(G)$ to the set $\mathbf{N}$ of positive integers such that $|c(u) - c(v)| + d(u, v) \geqslant 2$ for every pair u, v of distinct vertices of G, where $d(u, v)$ denotes the distance between u and v (the length of a shortest u–v path). For a positive integer k, with $1 \leqslant k \leqslant d$, a radio k-coloring of G is a function $c : V(G) \to \mathbf{N}$ that satisfies $|c(u)-c(v)|+d(u, v) \geqslant k+1$ for every pair u, v of distinct vertices of G. Thus 1-radio colorings and standard colorings are synonymous; while at the other extreme, radio d-colorings have been called radio labelings. Radio k-colorings were inspired by (FM Radio) Channel Assignment Problem (see [8–10], for example), which deals with optimal assignments of channels for radio stations. Radio k-colorings and radio labelings of graphs were studied in [4] and [2], respectively. In a radio labeling of a graph G, all vertices are required to be colored differently; while in a radio $(d - 1)$-coloring, antipodal vertices can be colored the same. For this reason, such colorings are also called antipodal colorings. These colorings have seen studied in [3]. Radio k-colorings of the class of paths P_n of order n were studied in [6]. Since the diameter P_n is $n - 1$, antipodal colorings of P_n satisfies $|c(u) - c(v)| + d(u, v) \geqslant n - 1$ for every pair u, v of distinct vertices of P_n.

While radio k-colorings of graphs G of order n concern the distances $d(u, v)$ between pairs u, v of distinct vertices of G and therefore paths of smallest length, much of the work concerning paths and cycles deals with those of greatest length. For distinct vertices u and v, let $D(u, v)$ denote the length of a longest u–v path. Of course, if G is a tree, then $D(u, v) = d(u, v)$ for every pair u, v of distinct vertices of G. Thus, antipodal colorings c of P_n can also be described as those satisfying

$$|c(u) - c(v)| + D(u, v) \geqslant n - 1 \tag{1}$$

for every two vertices u and v of P_n. This gives rise to functions c satisfying (1) for arbitrary connected graphs G of order n.

A hamiltonian coloring c of a connected graph G of order n is a function $c : V(G) \to \mathbf{N}$ for which $|c(u) - c(v)| + D(u, v) \geqslant n - 1$ for every pair u, v of distinct vertices of G. (Radio k-colorings and hamiltonian colorings of connected graphs are special cases of a more general graph coloring discussed in [5].) If c is a hamiltonian coloring of a connected graph G and u and v are two distinct vertices of G with $c(u) = c(v)$, then G contains a hamiltonian u–v path. For a hamiltonian coloring c, $\mathrm{hc}(c)$ denotes the largest color assigned to any vertex of G; while the hamiltonian chromatic number $\mathrm{hc}(G)$ is the minimum value of $\mathrm{hc}(c)$ over all hamiltonian colorings c of G. Hence $\mathrm{hc}(G)=1$ if and only if G is hamiltonian-connected. Thus the hamiltonian chromatic number of a connected graph G can be thought of as a measure of how close G is to being hamiltonian-connected, namely, the closer $\mathrm{hc}(G)$

is to 1, the closer G is to being hamiltonian-connected. If $\mathrm{hc}(c)=\mathrm{hc}(G)$, then c is a *minimum hamiltonian coloring*.

Two hamiltonian colorings c_1 and c_2 of a graph G are shown in Fig. 1. Consequently, $\mathrm{hc}(c_1) = 7$ and $\mathrm{hc}(c_2) = 6$. Indeed, it can be verified that c_2 is a minimum hamiltonian coloring and so the hamiltonian chromatic number of G is 6.

The concept of hamiltonian colorings of graph was introduced in [7], where it was shown that if G is a connected graph of order $n \geqslant 2$, then $\mathrm{hc}(G) \leqslant (n-2)^2 + 1$ and this bound is sharp. Also, for every two integers k and n with $1 \leqslant k \leqslant n-2$, there is a hamiltonian graph of order n with hamiltonian chromatic number k. Hamiltonian chromatic numbers of several well known graphs were established, including complete bipartite graphs, cycles, and Petersen graph.

To be sure, if G is a nonhamiltonian graph of order $n \geqslant 3$, then G is not hamiltonian-connected since for every pair u, v of adjacent vertices, G does not contain a hamiltonian u–v path. On the other hand, if u and v are nonadjacent vertices of G, then G may contain a hamiltonian u–v path. For such a graph then, $D(u, v) \leqslant n-2$ if u and v are adjacent and $D(u, v) \leqslant n-1$ if u and v are not adjacent. We define a connected graph G of order $n \geqslant 3$ to be *semihamiltonian-connected* if

$$D(u,v) = \begin{cases} n-2 & \text{if } uv \in E(G), \\ n-1 & \text{if } uv \notin E(G). \end{cases}$$

Now, let c be a hamiltonian coloring of a semihamiltonian-connected graph G order $n \geqslant 3$. Then $|c(u) - c(v)| + D(u, v) \geqslant n - 1$ for every pair u, v of distinct vertices of G. Hence if u and v are adjacent, then $|c(u) - c(v)| \geqslant 1$; while if u and v are not adjacent, then $|c(u) - c(v)| \geqslant 0$. That is, two vertices must be assigned distinct colors if the vertices are adjacent and may be assigned the same color if they are not adjacent. In other words, every hamiltonian coloring of a semihamiltonian-connected graph G of order $n \geqslant 3$ is an ordinary coloring of G and so $\mathrm{hc}(G) = \chi(G)$. Thus we have the following.

Proposition 1.1. *If G is a semihamiltonian-connected graph of order $n \geqslant 3$, then*

$$\mathrm{hc}(G) = \chi(G).$$

The graph P_3 and the Petersen graph are semihamiltonian-connected and so their hamiltonian chromatic number equals their chromatic number, which is 2 and 3, respectively. Whether there are other semihamiltonian-connected graphs is not known. If G is a connected nonhamiltonian graph of order $n \geqslant 3$ such that G has a hamiltonian u–v path for every pair u, v of nonadjacent vertices, then G need not be semihamiltonian-connected.

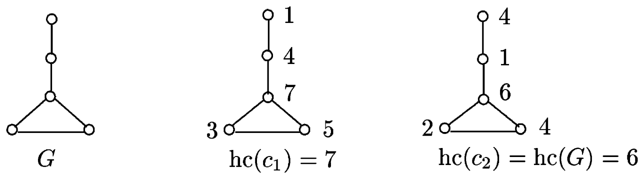

Fig. 1. Hamiltonian colorings of a graph.

For example, for $1 \leqslant m \leqslant n - m - 1$, the graph $G = K_1 + (K_m \cup K_{n-m-1})$ has this property but is not semihamiltonian-connected. On the other hand, the graph $G = K_{r,r}$, $r \geqslant 2$, with $n = 2r$, has the property that

$$D(u, v) = \begin{cases} n-1 & \text{if } uv \in E(G), \\ n-2 & \text{if } uv \notin E(G). \end{cases}$$

Thus, two vertices of $G = K_{r,r}$ must be assigned distinct colors in any hamiltonian coloring if they are not adjacent and may be assigned the same color if they are adjacent, that is, $\mathrm{hc}(G) = \chi(\overline{G}) = r$. The graph P_3, the Petersen graph, and $K_{r,r}$, $r \geqslant 2$, have the property that the numbers $D(u, v)$ have two distinct values, one if u and v are adjacent and another if u and v are not adjacent. For each of these graphs G of order n, one of the values of $D(u, v)$ is $n - 1$ and the other is $n - 2$.

2. On the circumference of graphs having many vertices with prescribed colors

Let c be a hamiltonian coloring of a connected graph G. For integers i and j with $1 \leqslant i \leqslant j \leqslant \mathrm{hc}(c)$, we define

$$V(c; i, j) = \{u \in V(G) : i \leqslant c(u) \leqslant j\}.$$

Let U be a set of vertices of G. If $|U| \geqslant 2$, then we define

$$\mathrm{dis}(c; U) = \min\{|c(u) - c(v)|\},$$

where the minimum is taken over all distinct pairs u, v of vertices in U. If $|U| \leqslant 1$, we define $\mathrm{dis}(c; U) = \mathrm{hc}(c)$. If $U = V(c; i, j)$, then we write $\mathrm{dis}(c; U) = \mathrm{dis}(c; i, j)$. More simply, we write

$$V(i, j) = V(c; i, j), \quad \mathrm{dis}(U) = \mathrm{dis}(c; U), \quad \text{and} \quad \mathrm{dis}(i, j) = \mathrm{dis}(c; i, j)$$

if the hamiltonian coloring c under discussion is clear.

The length of a longest cycle in a connected graph is called the *circumference* of G and is denoted by $\mathrm{cir}(G)$. If G is a tree, then we write $\mathrm{cir}(G) = 0$. For a hamiltonian coloring of a connected graph G of order n, we now show that if the sets $V(i, j)$ are sufficiently large (as a function of n and $\mathrm{dis}(i, j)$), then $\mathrm{cir}(G)$ is large as well. First, we present a lemma.

Lemma 2.1. *Let G be a connected graph of order $n \geqslant 3$, let c be a hamiltonian coloring of G, and let k be an integer with $0 \leqslant k \leqslant n - 3$. Assume that $\mathrm{cir}(G) < n - k$. Then $V(i, i + k)$ is an independent set in G for every integer i with $1 \leqslant i \leqslant \mathrm{hc}(c) - k$.*

Proof. Let i be an integer with $1 \leqslant i \leqslant \mathrm{hc}(c) - k$. Since $\mathrm{cir}(G) < n - k$, it follows that $D(u, v) \leqslant n - k - 2$ for every pair u, v of adjacent vertices of G. Since c is a hamiltonian coloring of G, it follows that $|c(u) - c(v)| \geqslant k + 1$ for every pair u, v of adjacent vertices of G. Moreover, $|c(u') - c(v')| \leqslant k$ for each pair u', v' of vertices in $V(i, i + k)$. Therefore, $V(i, i + k)$ is an independent set in G. $\square$

Theorem 2.2. *Let G be a connected graph of order* $n \geqslant 3$, *let c be a hamiltonian coloring of G, and let* i, j *be a pair of integers with* $1 \leqslant i \leqslant j \leqslant \mathrm{hc}(c)$ *and* $j - i \leqslant n - 3$. *If*

$$|V(i, j)| \geqslant \frac{n + \mathrm{dis}(i, j) + 2}{2},$$

then $\mathrm{cir}(G) \geqslant n - (j - i)$.

Proof. Assume that $\mathrm{cir}(G) \leqslant n-(j-i)-1$. If $|V(i, j)| \leqslant 1$, then $|V(i, j)| < (n+\mathrm{dis}(i, j)+2)/2$, a contradiction. Hence we may assume that $|V(i, j)| \geqslant 2$. Since $\mathrm{cir}(G) < n-(j-i)$, by virtue of Lemma 2.1, the set $V(i, j)$ is an independent set in G. Now, let $W = V(G) - V(i, j)$. Since $|V(i, j)| \geqslant 2$, there exist distinct vertices x and y in $V(i, j)$ with $|c(x)-c(y)| = \mathrm{dis}(i, j)$ and so $D(x, y) \geqslant n - 1 - \mathrm{dis}(i, j)$. Hence there exists an x–y path P containing at least $n-\mathrm{dis}(i, j)$ vertices of G. On the other hand, since $V(i, j)$ is independent in G, the vertices x and y are in $V(i, j)$, and P contains at most $|W|$ vertices that are not in $V(i, j)$, it follows that P contains at most $|W| + 1$ vertices of $V(i, j)$. Consequently, P contains at most $2|W| + 1$ vertices. Thus $n - \mathrm{dis}(i, j) \leqslant 2|W| + 1$, which implies that $|V(i, j)| < (n+\mathrm{dis}(i, j)+2)/2$, a contradiction. □

We are now able to provide a sufficient condition for a graph G to be hamiltonian.

Corollary 2.3. *Let G be a connected graph of order* $n \geqslant 3$. *If there exists a hamiltonian coloring of G such that at least* $(n + 2)/2$ *vertices of G are colored the same, then G is hamiltonian.*

Proof. Let c be a hamiltonian coloring of G such that at least $(n + 2)/2$ vertices of G are colored the same, say i. Then

$$|V(i, i)| \geqslant \frac{n + 2}{2} = \frac{n + \mathrm{dis}(i, i) + 2}{2}.$$

It then follows from Theorem 2.2 that $\mathrm{cir}(G) \geqslant n$ and so G is hamiltonian. □

To see that Corollary 2.3 cannot be improved, consider the graph $G = K_{r,r+1}$, where $r \geqslant 2$, with partite sets V_1 and V_2 such that $|V_1| = r$ and $|V_2| = r + 1$. Then G has order $n = 2r + 1$ and

$$D(u, v) = \begin{cases} 2r - 2 & \text{if } u, v \in V_1, \\ 2r - 1 & \text{if } uv \in E(G), \\ 2r & \text{if } u, v \in V_2. \end{cases}$$

Observe that a coloring c is a hamiltonian coloring of G if and only if

$$|c(u) - c(v)| \geqslant \begin{cases} 2 & \text{if } u, v \in V_1, \\ 1 & \text{if } uv \in E(G), \\ 0 & \text{if } u, v \in V_2. \end{cases}$$

Let $V_1 = \{v_1, v_2, \ldots, v_r\}$. Define a hamiltonian coloring c of G by $c(u) = 1$ for all $u \in V_2$ and $c(v_i) = 2i$ for $1 \leqslant i \leqslant r$. Then exactly $r + 1 = (n + 1)/2$ vertices of G are colored the same, but G is not hamiltonian.

Corollary 2.4. *Let G be a connected graph of order $n \geqslant 4$. If there exist a hamiltonian coloring c of G and an integer i with $1 \leqslant i < \mathrm{hc}(c)$ such that at least $(n + 2)/2$ vertices of G are colored i or $i + 1$, then $\mathrm{cir}(G) \geqslant n - 1$.*

Proof. If there exist a hamiltonian coloring c of G and an integer i with $1 \leqslant i < \mathrm{hc}(c)$ such that at least $(n + 2)/2$ vertices of G are colored by i or $i + 1$, then $|V(i, i + 1)| \geqslant \max\{3, (n + 2)/2\}$ and therefore, $\mathrm{dis}(i, i + 1) = 0$. It then follows from Theorem 2.2 that $\mathrm{cir}(G) \geqslant n - 1$. $\square$

We now present another lower bound for the circumference of a connected graph.

Theorem 2.5. *Let G be a connected graph of order $n \geqslant 3$ and let k be an integer such that $0 \leqslant k \leqslant n - 3$. If there exists a hamiltonian coloring c of G such that*

(a) *the sets $V(1,\ k + 1)$ and $V(\mathrm{hc}(c) - k,\ \mathrm{hc}(c))$ form a partition of $V(G)$, and*
(b) *there exists $U \in \{V(1,\ k + 1), V(\mathrm{hc}(c) - k,\ \mathrm{hc}(c))\}$ such that $|U| \geqslant 2$ and*

$$|U| \leqslant \frac{n - \mathrm{dis}(U)}{2}, \tag{2}$$

then $\mathrm{cir}(G) \geqslant n - k$.

Proof. Let $W = V(G) - U$. We wish to prove that $\mathrm{cir}(G) \geqslant n - k$. Assume, to the contrary, that $\mathrm{cir}(G) < n - k$. By Lemma 2.1, the sets U and W are independent in G. Since U and W are disjoint and $V(G) = U \cup W$, it follows that G is a bipartite graph with partite sets U and W. Since $|U| \geqslant 2$, there exist two distinct vertices $u, v \in U$ such that $|c(u) - c(v)| = \mathrm{dis}(U)$ and so $D(u, v) \geqslant n - 1 - \mathrm{dis}(U)$. Thus G contains a u–v path P of length at least $n - 1 - \mathrm{dis}(U)$ and so at most $\mathrm{dis}(U)$ vertices of G do not belong to P. Since G is bipartite with partite sets U and W and $u, v \in U$, there exists an integer j with $2 \leqslant j \leqslant |U|$ such that P contains exactly j vertices of U and exactly $j - 1$ vertices of W. Thus $2|U| - 1 \geqslant 2j - 1 \geqslant n - \mathrm{dis}(U)$. This means that $|U| \geqslant (n + 1 - \mathrm{dis}(U))/2$, which contradicts (2). Therefore, $\mathrm{cir}(G) \geqslant n - k$. $\square$

3. On the circumference and color sequences of graphs

For a hamiltonian coloring c of a connected graph G, let $\mathscr{C}$ be the set of all colors assigned to the vertices of G, that is, $\mathscr{C} = \{c(v) : v \in V(G)\}$. If $\mathscr{C} = \{c_1, c_2, \ldots, c_p\}$, where $c_1 < c_2 < \cdots < c_p = \mathrm{hc}(c)$, then $\mathrm{Seq}(c) = (c_1, c_2, \ldots, c_p)$ is called the *color sequence of* c. Similarly, as in [7], a set $S = \{u, v\}$ of two distinct vertices of G is called a *c-pair* if $c(u) = c(v)$. We define $c(S) = c(u) = c(v)$. A set $S = \{u, v\}$ of two distinct vertices of G is called a *c-semi pair* if $|c(u) - c(v)| \leqslant 1$. For integers a and b with $a \leqslant b$, the integer interval $[a\ ..\ b]$ is defined as $\{x \in \mathbf{Z}\ :\ a \leqslant x \leqslant b\}$.

Theorem 3.1. *For a connected graph G of order $n \geqslant 4$ and an integer k with $0 \leqslant k \leqslant n-3$, let c be a hamiltonian coloring of G with* $\mathrm{Seq}(c) = (c_1, c_2, \ldots, c_p)$, *where $p \geqslant 2$, such that*

$$\mathscr{C} \subseteq [c_1 \;..\; c_1 + k] \cup [c_p - k \;..\; c_p]. \tag{3}$$

If at least one of the three conditions

(a) $k = 0$;
(b) $c_p - k \leqslant c_1 + k$ *and* $\mathscr{C} \cap [c_p - k \;..\; c_1 + k]$ *is nonempty;*
(c) *there exist c-semipairs S and S', at least one which is a c-pair, such that the colors of the vertices of S are at most $c_1 + k$ and the colors of the vertices of S' are at least $c_p - k$, is satisfied, then* $\mathrm{cir}(G) \geqslant n - k$.

Proof. We may assume, without loss of generality, that $c_1 = 1$. Since $p \geqslant 2$, it follows that $c_p > 1$. Define $V_1 = V(1, k+1)$, $V_2 = V(c_p - k, c_p)$, $W_1 = V(1,1)$, $W_2 = V(c_p, c_p)$. Thus W_1 and W_2 are nonempty, as are V_1 and V_2. Moreover, if (a) holds, then $V_1 = W_1$ and $V_2 = W_2$. More generally, $W_i \subseteq V_i$ for $i = 1, 2$ and $V_1 \cup V_2 = V(G)$ by (3).

We wish to prove that $\mathrm{cir}(G) \geqslant n - k$. Assume, to the contrary, that $\mathrm{cir}(G) < n - k$. By Lemma 2.1, V_1 and V_2 are independent sets in G. Since $V_1 \cup V_2 = V(G)$, it follows that $V_1 \cap V_2$ is a set of isolated vertices of G. However, since G is a nontrivial connected graph, G has no isolated vertices and so $V_1 \cap V_2 = \emptyset$. Thus condition (b) does not hold, implying that at least one of conditions (a) and (c) holds.

Since $V(G)$ is partitioned into the independent sets V_1 and V_2, it follows that G is a bipartite graph with partite sets V_1 and V_2. Because $n \geqslant 4$, it follows that if (a) holds, then $|W_1| \geqslant 2$ or $|W_2| \geqslant 2$ and so either V_1 or V_2 contains a c-pair. On the other hand, if (c) holds, then V_1 or V_2 contains a c-pair. In either case, at least one of V_1 and V_2 contains a c-pair. Let $\{i, j\} = \{1, 2\}$ such that V_j contains a c-pair, say $\{x, y\}$. Since c is a hamiltonian coloring, $D(x, y) = n - 1$ and so there exists a hamiltonian x–y path in G. Since $x, y \in V_j$ and G is a bipartite graph with partite sets V_i and V_j, we have $|V_j| = |V_i| + 1$, which implies that $D(x', y') \leqslant n - 3$ for every pair x', y' of distinct vertices in V_i. Thus V_i contains no c-pair. Since $n \geqslant 4$, it follows that $|V_i| \geqslant 2$. Therefore, $V_i \neq W_i$ and (a) does not hold. Hence (c) holds. Consequently, V_i contains a c-semipair, say $\{x^*, y^*\}$. Since $|c(x^*) - c(y^*)| \leqslant 1$, we have $D(x^*, y^*) \geqslant n - 2$, which is a contradiction. □

Let G be a connected graph of order $n \geqslant 4$, let k be an integer with $0 \leqslant k \leqslant n - 3$, and let c be a hamiltonian coloring of G. Now, suppose that the sets $V_1 = V(1, k+1)$ and $V_2 = V(\mathrm{hc}(c) - k, \mathrm{hc}(c))$ form a partition of $V(G)$, where, say $|V_1| \geqslant |V_2|$. Thus $|V_1| \geqslant n/2$. Suppose that we wish to apply Theorem 3.1 to such a graph G. If the set V_1 contains a c-pair, so that $\mathrm{dis}(1, k+1) = 0$, and $|V_1| \geqslant (n+2)/2$, then $\mathrm{cir}(G) \geqslant n - k$ by Theorem 2.2. If, on the other hand, V_1 does not contain a c-pair but contains a c-semipair , so that $k \geqslant 1$ and $\mathrm{dis}(1, k+1) = 1$, and $|V_1| \geqslant (n+3)/2$, then $\mathrm{cir}(G) \geqslant n - k$ by Theorem 2.2. Hence to apply Theorem 3.1 to a graph G satisfying the conditions described above, we need only deal with the situation where $n/2 \leqslant |V_1| \leqslant (n+2)/2$.

We have already noted that if some hamiltonian coloring assigns the same color, namely 1, to every vertex in a connected graph G of order $n \geqslant 3$, then G is hamiltonian-connected.

By Theorem 3.1(a), if there exists a hamiltonian coloring that assigns one of two colors to every vertex of G, then G is hamiltonian.

Corollary 3.2. *Let G be a connected graph of order* $n \geqslant 4$. *If there exists a hamiltonian coloring c of G such that* $\mathrm{Seq}(c) = 1$ *or* $\mathrm{Seq}(c) = (1, r)$ *for some* $r \geqslant 2$, *then G is hamiltonian.*

Bondy and Chvátal [1] introduced the *closure* of a graph G of order $n \geqslant 3$ as the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n (in the resulting graph at each stage) until no such pair remains. The following theorem and corollary are due to Bondy and Chvátal.

Theorem A. *A graph is hamiltonian if and only its closure is hamiltonian.*

Corollary B. *If the closure of a graph G of order at least* 3 *is complete, then G is hamiltonian.*

Thus Corollary B gives a sufficient condition for a graph to be hamiltonian. Let G_0 be a hamiltonian-connected graph of even order $n = 2k \geqslant 6$. Then G_0 contains a hamiltonian cycle $u_1, v_1, u_2, v_2, \ldots, u_k, v_k, u_1$. We construct a new graph G from G_0 and k pairwise vertex-disjoint complete graphs of order $\ell \geqslant 3$, which we denote by $F_1, F_2, \ldots, F_k$, by identifying an edge of F_i with the edge $u_i v_i$ for each i $(1 \leqslant i \leqslant k)$. The graph G has order $k\ell$ but it is not hamiltonian-connected, as there is no hamiltonian u_i–v_i path for any i $(1 \leqslant i \leqslant k)$. On the other hand, there is a hamiltonian coloring of G with two colors, namely, assign u_i $(1 \leqslant i \leqslant k)$ the color $\ell - 1$ and assign all other vertices of G the color 1. By the remark above, G is hamiltonian. We now consider the Bondy and Chvátal closure of this graph G. Let $x \in V(F_i) - \{u_i, v_i\}$ $(1 \leqslant i \leqslant k)$ and $y \notin V(F_i)$ be nonadjacent vertices in G. Then $\deg_G x = \ell - 1$ and $\deg_G y \leqslant (2k-1) + (\ell - 1) - 1$. So

$$\deg_G x + \deg_G y \leqslant 2k + (2\ell - 4) = k\ell - (k-2)(\ell - 2) < k\ell,$$

which implies that no vertex in $V(F_i) - \{u_i, v_i\}$ can be adjacent to a vertex in $V(F_i)$ in the formation of the closure of G. Thus the closure of G is not complete and, even though Corollary 3.2 shows that G is hamiltonian, Corollary B does not.

The closure of the complete bipartite graph $K_{r,r}$, $r > 2$, is complete, however. Therefore, by Corollary B, $K_{r,r}$ is hamiltonian. On the other hand, there is no hamiltonian coloring of $K_{r,r}$ that assigns one of two colors to each of its vertices. Hence $K_{r,r}$ cannot be shown to be hamiltonian with the aid of Corollary 3.2. Therefore, Corollaries 3.2 and B are independent.

The next result gives a sufficient condition for a connected graph of order $n \geqslant 5$ to contain a cycle of length $n - 1$ or n.

Corollary 3.3. *Let G be a connected graph of order* $n \geqslant 5$. *If there exists a hamiltonian coloring c of G with* $\mathrm{hc}(c) \geqslant 4$ *satisfying one of the following conditions*:

(1) $\mathrm{Seq}(c) = (1, 2, \mathrm{hc}(c) - 1, \mathrm{hc}(c))$;
(2) $\mathrm{Seq}(c) = (1, \mathrm{hc}(c) - 1, \mathrm{hc}(c))$ *and there exists a c-pair S with* $c(S) = 1$;
(3) $\mathrm{Seq}(c) = (1, 2, \mathrm{hc}(c))$ *and there exists a c-pair S with* $c(S) = \mathrm{hc}(c)$; *then* $\mathrm{cir}(G) \geqslant n - 1$.

The following result shows that, in general, the hamiltonian chromatic number and the circumference cannot both be small.

Theorem 3.4. *If G is a connected graph of order* $n \geqslant 4$ *with* $2 \leqslant \mathrm{hc}(G) \leqslant n-1$, *then*

$$\mathrm{cir}(G) + \mathrm{hc}(G) \geqslant n + 2.$$

Proof. Let c be a minimum hamiltonian coloring of G with color set $\mathscr{C}$. Then 1 and $\mathrm{hc}(c)$ belong to $\mathscr{C}$, and $\mathrm{hc}(c) = \mathrm{hc}(G)$. If $\mathscr{C} = \{1, \mathrm{hc}(c)\}$, then $\mathrm{cir}(G) = n$ by Corollary 3.2 and therefore, $\mathrm{cir}(G) \geqslant n + 2 - \mathrm{hc}(G)$. So we may assume that $\mathscr{C} \neq \{1, \mathrm{hc}(c)\}$. Then $2 \leqslant \mathrm{hc}(c) - 1$ and the set $\mathscr{C} \cap [2..\mathrm{hc}(c) - 1]$ is nonempty. It then follows from Theorem 3.1 (b) that $\mathrm{cir}(G) \geqslant n + 2 - \mathrm{hc}(G)$ again. $\square$

One consequence of Theorem 3.4 is the following.

Corollary 3.5. *Let G be a connected graph of order* $n \geqslant 4$. *If* $\mathrm{hc}(G) = 2$, *then G is hamiltonian. If* $\mathrm{hc}(G) = 3$, *then* $\mathrm{cir}(G) \geqslant n - 1$.

The inequality of Theorem 3.4 is also sharp for $\mathrm{hc}(G) = 3$ since the Petersen graph has hamiltonian chromatic number 3, order 10, and circumference 9.

4. Hamiltonian colorings of trees

It was shown in [7] that $\mathrm{hc}(T) \leqslant (n-2)^2 + 1$ for every tree T of order $n \geqslant 2$. Furthermore, this bound is sharp for $n \geqslant 3$ since $\mathrm{hc}(K_{1,n-1}) = (n-2)^2 + 1$. For connected graphs that are not stars, an improved upper bound for the hamiltonian chromatic number in terms of its order exists, which we present in this section. Before presenting this bound, however, we need an additional definition. Let G be a connected graph of order $n \geqslant 4$. A sequence $v_1, v_2, \ldots, v_n$ of the vertices of G is called a *complementary hamiltonian* sequence (or simply *a* ch-*sequence*) of G if $D(v_i, v_{i+1}) \geqslant 2$ for all i with $1 \leqslant i < n$, and there exists j with $1 \leqslant j < n$ such that $D(v_j, v_{j+1}) \geqslant 3$. Observe that a ch-sequence in a tree is a hamiltonian path in its complement. We first present a lemma.

Lemma 4.1. *If T is a tree of order* $n \geqslant 4$ *such that T is not a star, then T contains a* ch-*sequence*.

Proof. We proceed by induction on the order n of T. If $n = 4$, then T is a path, say a path v_1, v_2, v_3, v_4. We see that v_2, v_4, v_1, v_3 is a ch-sequence of T. For $n \geqslant 5$, consider a peripheral vertex u of T. There are two cases.

Case 1: $T - u$ *is a star*. Let v denote the central vertex of $T - u$, and let $u_1, u_2, \ldots, u_{n-2}$ denote the vertices of degree one in $T - u$. Without loss of generality, we assume that u and u_{n-2} are adjacent in T. Then $v, u, u_1, \ldots, u_{n-2}$ is a ch-sequence of T.

Case 2: $T - u$ *is not a star*. By the induction hypothesis, there exists a ch-sequence $v_1, v_2, \ldots, v_{n-1}$ of $T - u$. If u and v_1 are adjacent in T, then $v_1, v_2, \ldots, v_{n-1}, u$ is a ch-sequence of T. Otherwise, $v, v_1, v_2, \ldots, v_{n-1}$ is a ch-sequence of T. $\square$

Theorem 4.2. *If T is a tree of order $n \geqslant 5$ such that T is not a star, then*

$$\mathrm{hc}(T) \leqslant (n-2)^2 - 1.$$

Furthermore, this bound is sharp.

Proof. By Lemma 4.1, T contains a ch-sequence $v_1, v_2, \ldots, v_n$ with $D(v_t, v_{t+1}) \geqslant 3$, where $1 \leqslant t < n$. Define a coloring c of T by

$$c(v_i) = \begin{cases} 1 + (i-1)(n-3) & \text{if } 1 \leqslant i \leqslant t, \\ (i-1)(n-3) & \text{if } t+1 \leqslant i \leqslant n. \end{cases}$$

Next, we show that c is a hamiltonian coloring of G. Let v_i and v_j be two distinct vertices of G, where $1 \leqslant i < j \leqslant n$. Observe that $|c(v_i) - c(v_j)| \geqslant (j-i)(n-3) - 1$. If $j - i \geqslant 2$, then $|c(v_i) - c(v_j)| \geqslant 2n - 7 \geqslant n - 2$ for all $n \geqslant 5$. If $j = i + 1$ and $i \neq t$, then $D(v_i, v_{i+1}) \geqslant 2$ and $|c(v_i) - c(v_j)| = n - 3$. If $j = i + 1$ and $i = t$, then $D(v_i, v_{i+1}) \geqslant 3$ and $|c(v_i) - c(v_j)| = n - 4$. In either case, $D(v_i, v_j) + |c(v_i) - c(v_j)| \geqslant n - 1$ and so c is a hamiltonian coloring of G. Since $\mathrm{hc}(c) = c(v_n) = (n-1)(n-3) = (n-2)^2 - 1$, it follows that $\mathrm{hc}(T) \leqslant (n-2)^2 - 1$.

To show that the upper bound is sharp, let T' be the tree of order $n \geqslant 5$ obtained by subdividing an edge of the star $K_{1,n-2}$. We claim that $\mathrm{hc}(T') = (n-2)^2 - 1$. Let $V(T') = \{v_1, v_2, \ldots, v_n\}$ and let c be a hamiltonian coloring of T'. Since $n \geqslant 5$, it follows that T' is not a path and so no two vertices of T' can be colored the same by any hamiltonian coloring of T'. Thus we may assume that $1 = c(v_1) < c(v_2) < \cdots < c(v_n)$. We first show that

$$\sum_{i=2}^{n} D(v_i, v_{i-1}) \leqslant 2n - 1. \tag{4}$$

Let $\mathscr{D} = \{D(v_i, v_{i-1}) : 2 \leqslant i \leqslant n\}$ denote a multiset. Observe that at most two numbers in $\mathscr{D}$ are 3 and the rest are 1 or 2. Let $u, v \in V(T')$ such that u is the central vertex of the star in T' and v is the end-vertex of T' with $D(u, v) = 2$ in T'. We consider two cases, according to whether $v_1 = u$ and $v_2 = v$ (equivalently, $v_n = u$ and $v_{n-1} = v$).

Case 1: $v_1 = u$ *and* $v_2 = v$. Then at most one number in $\mathscr{D}$ is 3 and the rest are 1 or 2. Thus $\sum_{i=2}^{n} D(v_i, v_{i-1}) \leqslant 3 + 2(n-2) = 2n - 1$.

Case 2: $v_1 \neq u$ *or* $v_2 \neq v$. Then there is an integer j with $2 \leqslant j \leqslant n$ such that $\{v_j, v_{j-1}\} = \{u, w\}$, where $w \in V(T') - \{u, v\}$, and so $D(v_j, v_{j-1}) = 1$. Since at most two numbers in $\mathscr{D} - \{d(v_j, v_{j-1})\}$ are 3 and the rest are 1 or 2, it follows that $\sum_{i=2}^{n} D(v_i, v_{i-1}) \leqslant 1 + 2 \cdot 3 + 2(n-4) = 2n - 1$. Hence (4) holds. Since $c(v_i) - c(v_{i-1}) \geqslant (n-1) - D(v_i, v_{i-1})$ for all i with $2 \leqslant i \leqslant n$, it then follows by (4) that

$$\begin{aligned} c(v_n) &\geqslant (n-1)^2 - \left(\sum_{i=2}^{n} D(v_i, v_{i-1}) \right) + c(v_1) \\ &\geqslant (n-1)^2 - (2n-1) + 1 = (n-2)^2 - 1. \end{aligned}$$

It follows that $\mathrm{hc}(T') \geqslant (n-2)^2 - 1$. Since $\mathrm{hc}(T) \leqslant (n-2)^2 - 1$ for all trees T of order $n \geqslant 5$ that are not stars, $\mathrm{hc}(T') = (n-2)^2 - 1$, as claimed. $\square$

It was shown in [7] that if T is a spanning tree of a connected graph G, then $\mathrm{hc}(G) \leqslant \mathrm{hc}(T)$. It is clear that if G is a connected graph of order at least 4 that is not a star, then G is spanned by a tree that is not a star. Thus the following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.3. *There exists no connected graph G of order* $n \geqslant 5$ *such that* $\mathrm{hc}(G)=(n-2)^2$. *Furthermore, if G is a connected graph of order* $n \geqslant 5$ *that is not a star, then* $\mathrm{hc}(G) \leqslant (n-2)^2 - 1$.

It was shown in [7] that $\mathrm{hc}(G) \leqslant (n-2)^2 + 1$ for every connected graph of order $n \geqslant 2$. The identity holds if and only if G is a star. It was also shown in [7] that if $n = 5$, then there exists no connected graph of order n with $\mathrm{hc}(G) = (n-2)^2$. Corollary 4.3 is an extension of this result for all $n \geqslant 5$.

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